Estimating real value from list of rounded numbers

W.Pasman, 25 dec 2014

A list of difference values is given $\{D_k, D_{k+1}, \ldots, D_{k+m}\}$ (m>0) where $D_n = \lfloor (n+1)R \rfloor - \lfloor nR \rfloor$ and R is a positive real number. Find a lower and upper bound for R.

Summing parts of the list

Write out the sum. Most elements in this sum cancel out:

$$\begin{aligned} & D_k + D_{k+1} + \ldots + D_{k+m} = \\ & = \left(\lfloor (k+1)R \rfloor - \lfloor kR \rfloor\right) + \left(\lfloor (k+2)R \rfloor - \lfloor (k+1)R \rfloor\right) + \\ & \ldots + \left(\lfloor (k+m+1)R \rfloor - \lfloor k+mR \rfloor\right) \\ & = -\lfloor kR \rfloor + \lfloor (k+m+1)R \rfloor = \lfloor (k+m+1)R \rfloor - \lfloor kR \rfloor \end{aligned} \tag{1}$$

Find lower and upper bounds for these terms:

$$\lfloor (k+m+1)R \rfloor - \lfloor kR \rfloor \geq \lfloor (m+1)R \rfloor + \lfloor kR \rfloor - \lfloor kR \rfloor = \lfloor (m+1)R \rfloor \tag{2}$$

and

$$\lfloor (k+m+1)R \rfloor - \lfloor kR \rfloor \leq \lfloor (m+1)R \rfloor + \lfloor kR \rfloor + 1 - \lfloor kR \rfloor = \lfloor (m+1)R \rfloor + 1 \tag{3}$$

so

$$\lfloor (m+1)R \rfloor \leq D_k + D_{k+1} + \ldots + D_{k+m} \leq \lfloor (m+1)R \rfloor + 1 \tag{4}$$

or, using $S_m = D_k + D_{k+1} + \ldots + D_{k+m-1}$ (notice m was lowered by 1) for any k:

$$\lfloor mR \rfloor \leq S_m \leq \lfloor mR \rfloor + 1 \tag{5}$$

In words, sum any m sequential elements from the list and you will always be just below or just above m*R.

Reversing the argument

Suppose R is unknown but S_m is given. We reverse the formula using that S_m is a natural number:

$$S_m \leq \lfloor mR \rfloor + 1 \Leftrightarrow \lfloor mR \rfloor \geq S_m - 1 \Leftrightarrow mR \geq S_m - 1 \Leftrightarrow R \geq \left(S_m - 1\right)/m \tag{6}$$

and

$$\lfloor mR \rfloor \leq S_m \Leftrightarrow \lfloor mR \rfloor < S_m + 1 \Leftrightarrow mR < S_m + 1 \Leftrightarrow R < \left(S_m + 1\right)/m \tag{7}$$

so

$$\frac{S_m - 1}{m} \leq R < \frac{S_m + 1}{m} \tag{8}$$

Example

Suppose the unknown R=8.7622. A random k is generated resulting in a difference list {8, 9, 9, 9, 9, 8, 9, 9, 9, 8, 9, 9, 9, 8, 9}.
The sum $S_m = 131$ and m=15, formula (8) gives

$$8.66667 = \frac{130}{15} \leq R < \frac{132}{15} = 8.8$$